# On Fall Colorings of Graphs

**Saeed Shaebani**
*Department of Mathematical Sciences*
*Institute for Advanced Studies in Basic Sciences (IASBS)*
*P.O. Box* 45195-1159*, Zanjan, Iran*
s_shaebani@iasbs.ac.ir

**Abstract**

A fall $k$-coloring of a graph $G$ is a proper $k$-coloring of $G$ such that each vertex of $G$ sees all $k$ colors on its closed neighborhood. We denote Fall($G$) the set of all positive integers $k$ for which $G$ has a fall $k$-coloring. In this paper, we study fall colorings of lexicographic product of graphs and categorical product of graphs and answer a question of [3] about fall colorings of categorical product of complete graphs. Then, we study fall colorings of union of graphs. Then, we prove that fall $k$-colorings of a graph can be reduced into proper $k$-colorings of graphs in a specified set. Then, we characterize fall colorings of Mycielskian of graphs. Finally, we prove that for each bipartite graph $G$, $\text{Fall}(G^c) \subseteq \{ \chi(G^c) \}$ and it is polynomial time to decision whether or not $\text{Fall}(G^c) = \{ \chi(G^c) \}$ .
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## 1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless and without multiple edges). Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$ and $[k] := \{i|\ i \in \mathbb{N},\ 1 \leq i \leq k\}$. A $k$-coloring (proper k-coloring) of $G$ is a function $f : V \to [k]$ such that for each $1 \leq i \leq k$, $f^{-1}(i)$ is an independent set. We say that $G$ is $k$-colorable whenever $G$ admits a $k$-coloring $f$, in this case, we denote $f^{-1}(i)$ by $V_i$ and call each $1 \leq i \leq k$, a color (of $f$) and each $V_i$, a color class (of $f$). The minimum integer $k$ for which $G$ has a $k$-coloring, is called the chromatic number of G and is denoted by $\chi(G)$.

Let $G$ be a graph, $f$ be a $k$-coloring of $G$ and $v$ be a vertex of $G$. The vertex $v$ is called colorful ( or color-dominating or $b$-dominating) if each color $1 \leq i \leq k$ appears on the closed neighborhood of $v$ ( $f(N[v]) = [k]$ ). The $k$-coloring $f$ is said to be a fall $k$-coloring (of $G$) if each vertex of $G$ is colorful. There are graphs $G$ for which $G$ has no fall $k$-coloring for any positive integer $k$. For example, $C_5$ ( a cycle with 5 vertices) and graphs with at least one edge and one isolated vertex, have not any fall $k$-colorings for any positive integer $k$. The notation Fall($G$) stands for the set of all positive integers $k$ for which $G$ has a fall $k$-coloring. Whenever $\text{Fall}(G) \neq \emptyset$, we call min(Fall($G$)) and max(Fall($G$)), fall chromatic number of $G$ and fall achromatic number of $G$ and denote them by $\chi_f(G)$ and $\psi_f(G)$, respectively. The terminology fall coloring was firstly introduced in 2000 in [3] and has received attention recently, see [1],[2],[3],[5].

## 2 Fall colorings of lexicographic product of graphs

Let $G$ and $H$ be graphs. The lexicographic product of G and H is defined the graph with vertex set $V(G) \times V(H)$ and edge set $\{ \ \{(x_1, y_1), (x_2, y_2)\} \mid x_1, x_2 \in V(G)$ and $y_1, y_2 \in V(H)$ and $[\ (\{x_1, x_2\} \in E(G))$ or $(x_1 = x_2, \{y_1, y_2\} \in E(H))\ ]\ \}$. For each $x \in V(G)$, the induced subgraph of $G[H]$ on $\{x\} \times V(H)$ is denoted by $H_x$.

Note that $G[H]$ and $H[G]$ are not necessarily isomorphic. For example, let $G := K_2$ and $H$ be the complement of $G$. $G[H]$ has 4 edges and $H[G]$ has 2 edges and therefore, they are not isomorphic. But lexicographic product of graphs is associative up to isomorphism ( For arbitrary graphs $G_1$, $G_2$ and $G_3$, $(G_1[G_2])[G_3]$ and $G_1[G_2[G_3]]$ are isomorphic.).

**Theorem 1.** *Let $G$ and $H$ be graphs and $k \in \mathrm{Fall}(G[H])$ and $f$ be a fall $k$-coloring of $G[H]$. Then, for each $x \in V(G)$, $S_x := f(V(H_x))$ forms a fall $|S_x|$-coloring of $H_x$.*

**Proof.** Let $x \in V(G)$ and $(x, y)$ be an arbitrary vertex of $H_x$ and its color be $\alpha$. Then, for each $\beta \in S_x \setminus \{\alpha\}$, there exists a vertex $(a, b)$ of $G[H]$ adjacent with $(x, y)$ which is colored $\beta$. Obviously $a = x$, otherwise, since $\beta \in S_x$, there exists a vertex $(x, z) \in V(H_x)$ colored $\beta$. $(x, y)$ is adjacent with $(a, b)$ and $x \neq a$, so $\{x, a\} \in E(G)$ and therefore, $(x, z)$ and $(a, b)$ are adjacent in $G[H]$ and both of them are colored $\beta$, which is a contradiction. Therefore, $a = x$ and $(a, b) \in V(H_x)$. Hence, $S_x$ forms a fall $|S_x|$-coloring of $H_x$. ■

**Corollary 1.** *Let $G$ and $H$ be graphs. Then,* $\mathrm{Fall}(G[H]) \neq \emptyset \Rightarrow \mathrm{Fall}(H) \neq \emptyset$*, or equivalently,* $\mathrm{Fall}(H) = \emptyset \Rightarrow \mathrm{Fall}(G[H]) = \emptyset$*.*

**Corollary 2.** *Let $G$ and $H$ be graphs such that* $\mathrm{Fall}(G[H]) \neq \emptyset$*. Then,* $\mathrm{Fall}(H) \neq \emptyset$ *and for each fall $k$-coloring $f$ of $G[H]$ and each $x \in V(G)$, $\chi_f(H) \leq |f(V(H_x))| \leq \psi_f(H)$.*

There are pairs of graphs $(G, H)$ for which $\mathrm{Fall}(G) = \emptyset$ but $\mathrm{Fall}(G[H]) \neq \emptyset$. For example, $\mathrm{Fall}(C_5) = \emptyset$ but $C_5[K_2]$ has a fall 5-coloring. First let's label the vertices of $C_5[K_2]$ lexicographically: $1 := (1, 1),\ 2 := (1, 2),\ 3 := (2, 1), \ldots,\ 10 := (5, 2)$. Here is a fall 5-coloring $f$ of $C_5[K_2]$: $f(1) = 1,\ f(2) = 2,\ f(3) = 3,\ f(4) = 4,\ f(5) = 1,\ f(6) = 5,\ f(7) = 2,\ f(8) = 4,\ f(9) = 5,\ f(10) = 3$. Also, there are pairs of graphs $(G, H)$ for which $\mathrm{Fall}(G) = \emptyset$ and $\mathrm{Fall}(H) \neq \emptyset$ and $\mathrm{Fall}(G[H]) = \emptyset$. For example, $\mathrm{Fall}(C_5) = \emptyset$ and $\mathrm{Fall}(K_1) \neq \emptyset$ and $\mathrm{Fall}(C_5[K_1]) = \mathrm{Fall}(C_5) = \emptyset$. The next theorem shows that if $\mathrm{Fall}(G) \neq \emptyset$ and $\mathrm{Fall}(H) \neq \emptyset$, then, $\mathrm{Fall}(G[H]) \neq \emptyset$.

**Theorem 2.** *Let $G$ and $H$ be graphs for which* $\mathrm{Fall}(G) \neq \emptyset$ *and* $\mathrm{Fall}(H) \neq \emptyset$*. Then,* $\{ \sum_{i=1}^{s} k_i \mid s \in \mathrm{Fall}(G),\ \forall 1 \leq i \leq s:\ k_i \in \mathrm{Fall}(H) \} \subseteq \mathrm{Fall}(G[H])$.

**Proof.** Let $s \in \mathrm{Fall}(G)$ and $g : V(G) \to [s]$ be a fall $s$-coloring of $G$ and for each $1 \leq i \leq s$, $k_i \in \mathrm{Fall}(H)$ and $h_i$ be a fall $k_i$-coloring of $H$. Let's color each vertex $(x, y)$ of $G[H]$ by color $(g(x), h_{g(x)}(y))$. Indeed, let's consider the function $f : V(G[H]) \to S := \{ (g(x), h_{g(x)}(y)) \mid (x, y) \in V(G) \times V(H) \}$ which assigns to

each $(x,y)$ of $G[H]$, $(g(x),h_{g(x)}(y))$. For each adjacent vertices $(x,y)$ and $(a,b)$ in $G[H]$, $\{x,a\}\in E(G)$ or ($x=a$ and $\{y,b\}\in E(H)$). So, $g(x)\neq g(a)$ or ($g(x)=g(a)$ and $h_{g(x)}(y)\neq h_{g(a)}(b)$). Therefore, $(g(x),h_{g(x)}(y))\neq(g(a),h_{g(a)}(b))$. This shows that $f$ is a $(\sum_{i=1}^{s}k_i)$-coloring of $G[H]$ such that uses exactly $\sum_{i=1}^{s}k_i$ colors. Now let's show that $f$ is a fall $(\sum_{i=1}^{s}k_i)$-coloring of $G[H]$. For each $(x,y)\in V(G[H])$ and each $(\alpha,\beta)\in S\setminus\{\ (g(x),h_{g(x)}(y))\ \}$, there is a vertex $(u,v)$ of $G[H]$ colored $(\alpha,\beta)$, or equivalently, $(g(u),h_{g(u)}(v))=(\alpha,\beta)$. Now, there are two cases:

Case I) The case that $g(x)=g(u)$. In this case, $h_{g(x)}=h_{g(u)}$ and $h_{g(x)}(y)\neq h_{g(u)}(v)$. Since $h_{g(x)}$ is a fall $k_{g(x)}$-coloring of $H$, there exists a vertex $z\in V(H)$ such that $\{z,y\}\in E(H)$ and $h_{g(x)}(z)=h_{g(u)}(v)$. The vertex $(x,z)$ of $G[H]$ is adjacent with $(x,y)$ and its color is $f((x,z))=(g(x),h_{g(x)}(z))=(g(u),h_{g(u)}(v))=(\alpha,\beta)$.

Case II) The case that $g(x)\neq g(u)$. Since $g$ is a fall $s$-coloring of $G$, there exists a vertex $z\in V(G)$ such that $\{x,z\}\in E(G)$ and $g(z)=g(u)$. So, $h_{g(u)}(v)=h_{g(z)}(v)$. The vertex $(z,v)$ is adjacent with $(x,y)$ in $G[H]$ and $f((z,v))=(g(z),h_{g(z)}(v))=(g(u),h_{g(u)}(v))=(\alpha,\beta)$.

Hence, $f$ is a fall $(\sum_{i=1}^{s}k_i)$-coloring of $G[H]$. Therefore, $\{\ \sum_{i=1}^{s}k_i\mid s\in\mathrm{Fall}(G),\ \forall 1\leq i\leq s\text{: }k_i\in\mathrm{Fall}(H)\ \}\subseteq\mathrm{Fall}(G[H])$. ■

**Corollary 3.** *Let $G$ and $H$ be graphs for which $\mathrm{Fall}(G)\neq\emptyset$ and $\mathrm{Fall}(H)\neq\emptyset$. Then, $\chi_f(G[H])\leq\chi_f(G)\chi_f(H)\leq\psi_f(G)\psi_f(H)\leq\psi_f(G[H])$.*

$\chi_f(G[H])$ and $\chi_f(G)\chi_f(H)$ are not necessarily equal. For example, $\chi_f(C_9)=3$ and $\chi_f(K_2)=2$. Therefore, $\chi_f(C_9)\chi_f(K_2)=6$, but $\chi_f(C_9[K_2])\leq 5$, first let's label the vertices of $C_9[K_2]$ lexicographically: 1:=(1,1), 2:=(1,2), 3:=(2,1), ..., 18:=(9,2). Here is a fall 5-coloring $f$ of $C_9[K_2]$: $f(1)=1,\ f(2)=4,\ f(3)=2,\ f(4)=3,\ f(5)=5,\ f(6)=1,\ f(7)=4,\ f(8)=2,\ f(9)=3,\ f(10)=1,\ f(11)=5,\ f(12)=2,\ f(13)=4,\ f(14)=3, f(15)=1,\ f(16)=2,\ f(17)=5,\ f(18)=3$. Also, $\psi_f(G)\psi_f(H)$ and $\psi_f(G[H])$ are not necessarily equal. For example, $\psi_f(C_8)=2$ and $\psi_f(K_2)=2$ and therefore, $\psi_f(C_8)\psi_f(K_2)=4$. But $\psi_f(C_8[K_2])\geq 5$. First let's label the vertices of $C_8[K_2]$ lexicographically: $1:=(1,1),\ 2:=(1,2),\ 3:=(2,1),\ldots,\ 16:=(8,2)$. Here is a fall 5-coloring $f$ of $C_8[K_2]$: $f(1)=1,\ f(2)=2,\ f(3)=3,\ f(4)=4,\ f(5)=5,\ f(6)=1,\ f(7)=2,\ f(8)=3,\ f(9)=4,\ f(10)=1,\ f(11)=5,\ f(12)=2,\ f(13)=3,\ f(14)=1,\ f(15)=5,\ f(16)=4$.

Theorem 2 says that if $G$ and $H$ are graphs for which $\mathrm{Fall}(G)\neq\emptyset$ and $\mathrm{Fall}(H)\neq\emptyset$, Then, $\{\ \sum_{i=1}^{s}k_i\mid s\in\mathrm{Fall}(G),\ \forall 1\leq i\leq s:\ k_i\in\mathrm{Fall}(H)\ \}\subseteq\mathrm{Fall}(G[H])$. Since $5\in\mathrm{Fall}(C_9[K_2])$ and $5\notin\{\ \sum_{i=1}^{s}k_i\mid\ s\in\mathrm{Fall}(C_9),\ \forall\ 1\leq i\leq s\text{: }k_i\in\mathrm{Fall}(K_2)\ \}$, $\mathrm{Fall}(G[H])$ and $\{\ \sum_{i=1}^{s}k_i\mid s\in\mathrm{Fall}(G),\ \forall\ 1\leq i\leq s:\ k_i\in\mathrm{Fall}(H)\}$ are not necessarily equal in this theorem.

**Theorem 3.** *There are pairs of graphs $(G,H)$ for which $\mathrm{Fall}(G)\neq\emptyset$ and $\mathrm{Fall}(H)\neq\emptyset$ and the following strictly inequality holds.*

*$\chi_f(G[H])<\chi_f(G)\chi_f(H)<\psi_f(G)\psi_f(H)<\psi_f(G[H])$.*

**Proof.** Set $G:=C_6\bigvee C_8\bigvee C_9$ ( the join of $C_6$ and $C_8$ and $C_9$) and $H:=K_2$. Since $(C_6\bigvee C_8\bigvee C_9)[K_2]$ and $(\ C_6[K_2]\ )\bigvee(\ C_8[K_2]\ )\bigvee(\ C_9[K_2]\ )$ are isomorphic, $\chi_f((C_6\bigvee C_8\bigvee C_9)[K_2])=\chi_f(C_6[K_2])+\chi_f(C_8[K_2])+\chi_f(C_9[K_2])\leq 4+4+5=13$ and $\psi_f((C_6\bigvee C_8\bigvee C_9)[K_2])=\psi_f(C_6[K_2])+\psi_f(C_8[K_2])+\psi_f(C_9[K_2])\geq 6+5+6=$

17 . Also, $\chi_f(C_6 \bigvee C_8 \bigvee C_9) = 7$ and $\psi_f(C_6 \bigvee C_8 \bigvee C_9) = 8$ and $\chi_f(K_2) = \psi_f(K_2) = 2$, as desired. $\blacksquare$

**Theorem 4.** *For each $\varepsilon > 0$, There exists a pair of graphs $(S, T)$ for which $\min\{\psi_f(S[T]) - \psi_f(S)\psi_f(T), \psi_f(S)\psi_f(T) - \chi_f(S)\chi_f(T), \chi_f(S)\chi_f(T) - \chi_f(S[T])\} \geq \varepsilon$.*

**Proof.** With no loss of generality, we can assume that $\varepsilon$ is a natural number. Set $G := C_6 \bigvee C_8 \bigvee C_9$ and $S := K_\varepsilon[G]$ and $T := K_2$. Since $S[T]$ and $K_\varepsilon[G[T]]$ are isomorphic and $\chi_f(K_\varepsilon[G[T]]) = \varepsilon\chi_f(G[T])$ and $\psi_f(K_\varepsilon[G[T]]) = \varepsilon\psi_f(G[T])$, the theorem implies. $\blacksquare$

One can easily observe that if $G$ and $H$ are graphs such that $\text{Fall}(G[H]) \neq \emptyset$, then, $\chi_f(G[H]) \geq \omega(G)\chi_f(H)$. The next clear proposition introduces a sufficient condition for equality.

**Proposition 1.** *Let $G$ and $H$ be graphs such that* $\text{Fall}(G) \neq \emptyset$ *and* $\text{Fall}(H) \neq \emptyset$ *and $\chi_f(G) = \omega(G)$. Then, $\chi_f(G[H]) = \chi_f(G)\chi_f(H) = \omega(G)\chi_f(H)$.*

**Corollary 4.** *If $G$ is a tree or a complete graph or $C_{2k}$ (for some $k \in \mathbb{N} \setminus \{1\}$) and $H$ is a graph such that* $\text{Fall}(H) \neq \emptyset$*, then, $\chi_f(G[H]) = \chi_f(G)\chi_f(H) = \omega(G)\chi_f(H)$.*

Corollary 1 says that in every fall $k$-coloring of $G[H]$ and each $x \in V(G)$, the number of colors appear on $V(H_x)$ is at most $\psi_f(H)$. Hence, $\psi_f(G[H]) \leq (\delta(G) + 1)\psi_f(H)$. The following clear proposition introduces a condition for equality.

**Proposition 2.** *Let $G$ and $H$ be graphs for which* $\text{Fall}(G) \neq \emptyset$ *and* $\text{Fall}(H) \neq \emptyset$ *and $\psi_f(G) = \delta(G) + 1$. Then, $\psi_f(G[H]) = \psi_f(G)\psi_f(H) = (\delta(G) + 1)\psi_f(H)$.*

**Corollary 5.** *If $G$ is a tree or a complete graph or $C_{3k}$ (for some $k \in \mathbb{N}$) and $H$ is a graph such that* $\text{Fall}(H) \neq \emptyset$*, then, $\psi_f(G[H]) = \psi_f(G)\psi_f(H) = (\delta(G) + 1)\psi_f(H)$.*

# 3 Type-II graph homomorphisms and lexicographic product of graphs

Now we study a type of graph homomorphisms that is related to fall colorings of graphs.

**Definition 1.** Let $G$ and $H$ be graphs. A function $f : V(G) \to V(H)$ is called a type-II graph homomorphism from $G$ to $H$ if $f$ satisfies the following two conditions.

1) $\{u, v\} \in E(G) \Rightarrow \{f(u), f(v)\} \in E(H)$.
2) $\{u_1, v_1\} \in E(H) \Rightarrow \forall v \in f^{-1}(v_1) : \exists u \in f^{-1}(u_1)$ s.t $\{u, v\} \in E(G)$. $\spadesuit$

Type-II graph homomorphisms introduced by Laskar and Lyle in 2009 in [5]. They showed that for any graph $G$, $k \in \text{Fall}(G)$ iff there exists a type-II graph homomorphism from $G$ to $K_k$. Note that every type-II graph homomorphism from

a graph $G$ to a complete graph, is surjective. If $f_1$ is a type-II graph homomorphism from $G$ to $H$ and $f_2$ is a type-II graph homomorphism from $H$ to $I$, then, $f_2 o f_1$ is a type-II graph homomorphism from $G$ to $I$. Also, if there exists a type-II graph homomorphism from $G$ to $H$ and $k \in \mathrm{Fall}(H)$, then, $k \in \mathrm{Fall}(G)$. If there exists a type-II graph homomorphism from $G_1$ to $G_2$ and a type-II graph homomorphism from $H_1$ to $H_2$, then, there exists a type-II graph homomorphism from $G_1 \square H_1$ to $G_2 \square H_2$. We prove a similar theorem for lexicographic product of graphs.

**Theorem 5.** *Let $G_1$, $G_2$, $H_1$ and $H_2$ be graphs and $f_1$ be a type-II graph homomorphism from $G_1$ to $G_2$ and $f_2$ be a surjective type-II graph homomorphism from $H_1$ to $H_2$. Then, there exists a type-II graph homomorphism $f_3$ from $G_1[H_1]$ to $G_2[H_2]$.*

**Proof.** Let $f_3 : V(G_1[H_1]) \rightarrow V(G_2[H_2])$ be defined the function which assigns to each $(g, h) \in V(G_1[H_1])$, $f_3((g, h)) = (f_1(g), f_2(h))$. For each $\{\ (x_1, y_1), (x_2, y_2)\ \} \in E(G_1[H_1])$, $\{x_1, x_2\} \in E(G_1)$ or $(x_1 = x_2$ and $\{y_1, y_2\} \in E(H_1))$. Therefore, $\{\ f_1(x_1), f_1(x_2)\ \} \in E(G_2)$ or $(f_1(x_1) = f_1(x_2)$ and $\{\ f_2(y_1), f_2(y_2)\ \} \in E(H_2))$. Hence, $\{\ (f_1(x_1), f_2(y_1)), (f_1(x_2), f_2(y_2))\ \} \in E(G_2[H_2])$ and consequently, the property 1 holds. Now for each $\{\ (\alpha_1, \beta_1), (\alpha_2, \beta_2)\ \} \in E(G_2[H_2])$ and each $(u_1, v_1) \in f_3^{-1}((\alpha_1, \beta_1))$, there are two cases:

Case I) The case that $\{\alpha_1, \alpha_2\} \in E(G_2)$. Since $f_1$ is a type-II graph homomorphism and $u_1 \in f_1^{-1}(\alpha_1)$, there exists $u_2 \in f_1^{-1}(\alpha_2)$ such that $\{u_1, u_2\} \in E(G_1)$. Surjectivity of $f_2$ implies that there exists $v_2 \in f_2^{-1}(\beta_2)$. Therefore, $(u_2, v_2) \in f_3^{-1}((\alpha_2, \beta_2))$ and $\{\ (u_1, v_1), (u_2, v_2)\ \} \in E(G_1[H_1])$ and accordingly, the property 2 holds.

Case II) The case that $\alpha_1 = \alpha_2$ and $\{\beta_1, \beta_2\} \in E(H_2)$. In this case, $u_1 \in f_1^{-1}(\alpha_2)$ and since $f_2$ is a type-II graph homomorphism and $v_1 \in f_2^{-1}(\beta_1)$, there exists $v_2 \in f_2^{-1}(\beta_2)$ such that $\{v_1, v_2\} \in E(H_1)$. Hence, $(u_1, v_2) \in f_3^{-1}((\alpha_2, \beta_2))$ and $\{\ (u_1, v_1), (u_1, v_2)\ \} \in E(G_1[H_1])$ and therefore, the property 2 holds. Thus, $f_3$ is a type-II graph homomorphism. ∎

**Corollary 6.** *If $G$ and $H$ are graphs such that $r_1 \in \mathrm{Fall}(G)$ and $r_2 \in \mathrm{Fall}(H)$, then*
$\chi_f(G[H]) \leq \chi_f(G[K_{r_2}]) \leq \chi_f(K_{r_1}[K_{r_2}]) \leq \psi_f(K_{r_1}[K_{r_2}]) \leq \psi_f(G[K_{r_2}]) \leq \psi_f(G[H])$.

# 4 Fall colorings of categorical product of graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The graph $G_1 \times G_2 := (V_1 \times V_2, \{\ \{\ (x_1, y_1), (x_2, y_2)\ \} \mid \{x_1, x_2\} \in E(G_1)$ and $\{y_1, y_2\} \in E(G_2)\ \})$ is called the categorical product of $G$ and $H$.

Categorical product of graphs is commutative and associative up to isomorphism (For each arbitrary graphs $G_1$, $G_2$ and $G_3$, $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic, also, $(G_1 \times G_2) \times G_3$ and $G_1 \times (G_2 \times G_3)$ are isomorphic.). For arbitrary graphs $G$ and $H$, if $E(G) = \emptyset$ or $E(H) = \emptyset$, then, $E(G \times H) = \emptyset$ and therefore, $G \times H$ has only a fall 1-coloring and $\mathrm{Fall}(G \times H) = \{1\}$. Thus, hereafter, let's focus on nonempty edge set graphs, unless stated otherwise. Firstly, note that $\mathrm{Fall}(G :=$

$( \{a, b, c, d\} \ , \ \{ \{a, b\}, \{b, c\}, \{c, a\}, \{d, a\} \ \} \ ) = \emptyset$ and $\text{Fall}(G \times G) = \emptyset$. Secondly, note that $\text{Fall}(C_5 := ( \ \{0, 1, 2, 3, 4\} \ , \ \{ \ \{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 0\} \ \} \ )) = \emptyset$, but the function $f : V(C_5 \times C_5) \to [5]$ which assigns to each $(i, j)$ of $V(C_5 \times C_5)$, $f((i, j)) :=$(the arithmetic residue of $i + 2j$ modulo 5)+1 where the last + is the natural summation in $\mathbb{Z}$, is a fall 5-coloring of $C_5 \times C_5$, and therefore, $\text{Fall}(C_5 \times C_5) \neq \emptyset$. The next theorem implies that if $\text{Fall}(G) \neq \emptyset$ or $\text{Fall}(H) \neq \emptyset$, then, $\text{Fall}(G \times H) \neq \emptyset$.

**Theorem 6.** *For each $n \in \mathbb{N}$ and each arbitrary graphs $G_1, \ldots, G_n$,*
$\forall 1 \leq i \leq n : \text{Fall}(G_i) \subseteq \text{Fall}(\times_{i=1}^{n} G_i)$.

**Proof.** Since categorical product of graphs is commutative and associative up to isomorphism, it suffices to prove that $\text{Fall}(G_1) \subseteq \text{Fall}(G_1 \times G_2)$. If $\text{Fall}(G_1) = \emptyset$, the theorem holds trivially. For each $k \in \text{Fall}(G_1)$, there exists a fall $k$-coloring $f$ of $G_1$. Now, the function $g : V(G_1 \times G_2) \to [k]$ which assigns to each $(u, v) \in V(G_1 \times G_2)$, $g((u, v)) = f(u)$ is a fall $k$-coloring of $G_1 \times G_2$ and therefore, $k \in \text{Fall}(G_1 \times G_2)$. Hence, $\text{Fall}(G_1) \subseteq \text{Fall}(G_1 \times G_2)$. ■

**Corollary 7.** *For each $n \in \mathbb{N}$ and each arbitrary graphs $G_1, \ldots, G_n$ such that for each $i \in [n]$, $\text{Fall}(G_i) \neq \emptyset$, the following inequalities hold.*
$\chi_f(\times_{i=1}^{n} G_i) \leq \min\{ \ \chi_f(G_i) \mid i \in [n] \ \} \leq \max\{ \ \psi_f(G_i) \mid i \in [n] \ \} \leq \psi_f(\times_{i=1}^{n} G_i)$.

Now again type-II graph homomorphisms:

**Theorem 7.** *Let $G_1$, $G_2$, $H_1$ and $H_2$ be graphs and $f_1$ be a type-II graph homomorphism from $G_1$ to $G_2$ and $f_2$ be a type-II graph homomorphism from $H_1$ to $H_2$. Then, there exists a type-II graph homomorphism $f_3$ from $G_1 \times H_1$ to $G_2 \times H_2$.*

**Proof.** Let $f_3 : V(G_1 \times H_1) \to V(G_2 \times H_2)$ be defined the function which assigns to each $(g, h) \in V(G_1 \times H_1)$, $f_3((g, h)) = (f_1(g), f_2(h))$. For each $\{ \ (x_1, y_1), (x_2, y_2) \ \} \in E(G_1 \times H_1)$, $\{x_1, x_2\} \in E(G_1)$ and $\{y_1, y_2\} \in E(H_1)$. Therefore, $\{ \ f_1(x_1), f_1(x_2) \ \} \in E(G_2)$ and $\{ \ f_2(y_1), f_2(y_2) \ \} \in E(H_2)$. Hence, $\{ \ f_3((x_1, y_1)), f_3((x_2, y_2)) \ \} \in E(G_2 \times H_2)$ and therefore, the property 1 of type-II graph homomorphisms holds. Now for each $\{ \ (a, b), (c, d) \ \} \in E(G_2 \times H_2)$ and each $(\alpha, \beta) \in f_3^{-1}( \ (c, d) \ )$, $\alpha \in f_1^{-1}(c)$ and $\beta \in f_2^{-1}(d)$. So, there exist $x \in f_1^{-1}(a)$ and $y \in f_2^{-1}(b)$ such that $\{x, \alpha\} \in E(G_1)$ and $\{y, \beta\} \in E(H_1)$, hence, $(x, y) \in f_3^{-1}( \ (a, b) \ )$ and $\{ \ (x, y), (\alpha, \beta) \ \} \in E(G_1 \times H_1)$. So, the property 2 of type-II graph homomorphisms holds, too. Consequently, $f_3$ is a type-II graph homomorphism.

■

We know that if $f$ is a type-II graph homomorphism from $G$ to $H$ and $k \in \text{Fall}(H)$, then, $k \in \text{Fall}(G)$. Also, for each graph $G$ and each natural number $k$, $k \in \text{Fall}(G)$ iff there exists a type-II graph homomorphism from $G$ to $K_k$. Therefore, the previous theorem implies the following corollary.

**Corollary 8.** *Let $n \in \mathbb{N}$ and for each $i \in [n]$, $G_i$ be a graph and $k_i \in \text{Fall}(G_i)$. Then, there exists a type-II graph homomorphism from $\times_{i=1}^{n} G_i$ to $\times_{i=1}^{n} K_{k_i}$ and*

$\mathrm{Fall}(\times_{i=1}^{n} K_{k_i}) \subseteq \mathrm{Fall}(\times_{i=1}^{n} G_i)$. *Also,* $\chi_f(\times_{i=1}^{n} G_i) \leq \chi_f(\times_{i=1}^{n} K_{k_i}) \leq \psi_f(\times_{i=1}^{n} K_{k_i}) \leq \psi_f(\times_{i=1}^{n} G_i)$. *These inequalities can easily extend to more inequalities in general. For example, in the case* $n = 2$, $\chi_f(G_1 \times G_2) \leq \left\{ \begin{array}{l} \chi_f(K_{k_1} \times G_2) \\ \chi_f(G_1 \times K_{k_2}) \end{array} \right. \leq \chi_f(K_{k_1} \times K_{k_2}) \leq \psi_f(K_{k_1} \times K_{k_2}) \leq \left\{ \begin{array}{l} \psi_f(K_{k_1} \times G_2) \\ \psi_f(G_1 \times K_{k_2}) \end{array} \right. \leq \psi_f(G_1 \times G_2)$.

Dunbar, et al. in [3] showed that for each $m, n \in \mathbb{N} \setminus \{1\}$, $\mathrm{Fall}(K_m \times K_n) = \{m, n\}$. They also showed that if $n \in \mathbb{N} \setminus \{1\}$ and for each $i \in [n]$, $r_i \in \mathbb{N} \setminus \{1\}$, then, $\{r_1, ..., r_n\} \subseteq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$. They constructed a fall 6-coloring of $K_2 \times K_3 \times K_4$ and asked for conditions for a finite and with more than two elements set $S := \{r_1, ..., r_n\} \subseteq \mathbb{N} \setminus \{1\}$ for which $S \subsetneqq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$.

**Theorem 8.** *Let* $n \geq 3$, $S := \{r_1, ..., r_n\} \subseteq \mathbb{N} \setminus \{1\}$, $r_1 < r_2 < ... < r_n$ *and* $S$ *contain at least one even integer. Then,* $S \subsetneqq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$, *besides,* $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ *contains an integer greater than* $r_n$.

**Proof.** There are five cases.

Case I) The case that $r_1 = 2$. In this case, let $t \in \{r_1, ..., r_n\} \setminus \{r_1, r_n\}$. Consider $K_2 \times K_t \times K_{r_n}$. Let $\sigma$ be a disarrangement of $[t]$ ( a permutation $\sigma$ of $[t]$ such that for each $i \in [t]$, $\sigma(i) \neq i$). Obviously, $\{ \{(1, i, 1), (1, \sigma(i), 2), (2, i, 2), (2, \sigma(i), 1)\} \mid 1 \leq i \leq t \} \bigcup \{ \{(x, y, z) \mid (x, y, z) \in K_2 \times K_t \times K_{r_n},\ z = i \} \mid 3 \leq i \leq r_n \}$ is the set of color classes of a fall $(r_n + t - 2)$-coloring of $K_2 \times K_t \times K_{r_n}$. But $r_n + t - 2 > r_n$ and therefore, in this case, $\mathrm{Fall}(K_2 \times K_t \times K_{r_n})$ contains an integer greater than $r_n$.

Case II) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains at least two distinct even integers such that one of them is $r_n$ and the other is $r_s$ that $s \in \{1, ..., n-1\}$. Let $r_j \in \{r_1, ..., r_n\} \setminus \{r_s, r_n\}$. Consider $K_{r_s} \times K_{r_j} \times K_{r_n}$ and a disarrangement $\sigma$ of $[r_j]$. For each $1 \leq t \leq r_j$, color the vertices $(1, t, 1), (1, \sigma(t), 2), (2, t, 2)$ and $(2, \sigma(t), 1)$ with the color $t$ and color each other vertex $(x, y, z)$ with the color $\lfloor \frac{x-1}{2} \rfloor (\frac{r_j r_n}{2}) + \lfloor \frac{z-1}{2} \rfloor r_j +$ the color of $(x - 2\lfloor \frac{x-1}{2} \rfloor, y, z - 2\lfloor \frac{z-1}{2} \rfloor)$. This is a fall $\frac{r_s r_j r_n}{4}$-coloring of $K_{r_s} \times K_{r_j} \times K_{r_n}$. Since $2 < r_1$, $\frac{r_s r_j r_n}{4} > \max\{r_s, r_j, r_n\}$. Hence, Theorem 6 implies that $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than $r_n$.

Case III) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains at least two distinct even integers such that none of them is $r_n$. Similar to the case II, $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than $r_n$.

Case IV) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains exactly one even integer and $r_n$ is even. In this case, consider $K_{r_{n-2}-1} \times K_{r_{n-1}} \times K_{r_n}$ and a disarrangement $\sigma$ of $[r_{n-1}]$. For each $1 \leq t \leq r_{n-1}$, color the vertices $(1, t, 1), (1, \sigma(t), 2), (2, t, 2)$ and $(2, \sigma(t), 1)$ with the color $t$ and color each other vertex $(x, y, z)$ of $K_{r_{n-2}-1} \times K_{r_{n-1}} \times K_{r_n}$ with the color $\lfloor \frac{x-1}{2} \rfloor (\frac{r_{n-1} r_n}{2}) + \lfloor \frac{z-1}{2} \rfloor r_{n-1} +$ the color of $(x - 2\lfloor \frac{x-1}{2} \rfloor, y, z - 2\lfloor \frac{z-1}{2} \rfloor)$. Also, color each vertex $(r_{n-2}, y, z)$ of $K_{r_{n-2}} \times K_{r_{n-1}} \times K_{r_n}$ with the color $\frac{(r_{n-2}-1) r_{n-1} r_n}{4} + 1$. Therefore, a fall $(\frac{(r_{n-2}-1) r_{n-1} r_n}{4} + 1)$-coloring of $K_{r_{n-2}} \times K_{r_{n-1}} \times K_{r_n}$ and also of $\times_{i=1}^{n} K_{r_i}$ yields. But, $\frac{(r_{n-2}-1) r_{n-1} r_n}{4} + 1 > r_n$. Thus, $\mathrm{Fall}(K_2 \times K_t \times K_{r_n})$ and therefore $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than $r_n$.

Case V) The case that $2 < r_1$ and $\{r_1, ..., r_n\}$ contains exactly one even integer and $r_n$ is odd. In this case, similar to the case IV, $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than $r_n$.

Accordingly, in all cases, $\{r_1, ..., r_n\} \subsetneqq \mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$. Besides, $\mathrm{Fall}(\times_{i=1}^{n} K_{r_i})$ contains an integer greater than $r_n$.

■

Even though Dunbar, et al. in [3] constructed a fall 6-coloring of $K_2 \times K_3 \times K_4$, this theorem also shows that in the corollary 7, the inequality $\max\{ \psi_f(G_i) \mid i \in [n] \} \leq \psi_f(\times_{i=1}^{n} G_i)$ can be strict in many cases. But we conjecture that the inequality $\chi_f(\times_{i=1}^{n} G_i) \leq \min\{ \chi_f(G_i) \mid i \in [n] \}$ is always an equality.

**Conjecture 1.** *For each $n \in \mathbb{N}$ and for each arbitrary graphs $G_1, \ldots, G_n$ such that for each $i \in [n]$, $\mathrm{Fall}(G_i) \neq \emptyset$, the following equality holds.*
$\chi_f(\times_{i=1}^{n} G_i) = \min\{\chi_f(G_i) |\ i \in [n]\}$.

# 5 Fall colorings of union of graphs

Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, $G_i$ be a graph. The graph $( \bigcup_{i=1}^{n} (\{i\} \times V(G_i))\ ,\ \bigcup_{i=1}^{n} \{ \{(i,x),(i,y)\} \mid \{x,y\} \in E(G_i) \} )$ is called the union graph of $G_1, ..., G_n$ and is denoted by $\biguplus_{i=1}^{n} G_i$.

The following obvious theorem describes fall colorings of union of graphs.

**Theorem 9.** *Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, $G_i$ be a graph. Then, the following three statements hold.*
*1) If $\mathrm{Fall}(\biguplus_{i=1}^{n} G_i) \neq \emptyset$, then, for each $1 \leq i \leq n$, $\mathrm{Fall}(G_i) \neq \emptyset$.*
*2) $\mathrm{Fall}(\biguplus_{i=1}^{n} G_i) = \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.*
*3) If $\mathrm{Fall}(\biguplus_{i=1}^{n} G_i) \neq \emptyset$, then, $\chi_f(\biguplus_{i=1}^{n} G_i) = \min \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$ and $\psi_f(\biguplus_{i=1}^{n} G_i) = \max \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.*

Since any graph $G$ is isomorphic to any union graph of all its connected components, the following corollary yields immediately.

**Corollary 9.** *Let $G$ be a graph and $G_i$ $(1 \leq i \leq n)$ be all its connected components. Then, the following three statements hold.*
*1) If $\mathrm{Fall}(G) \neq \emptyset$, then, for each $1 \leq i \leq n$, $\mathrm{Fall}(G_i) \neq \emptyset$.*
*2) $\mathrm{Fall}(G) = \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.*
*3) If $\mathrm{Fall}(G) \neq \emptyset$, then, $\chi_f(G) = \min \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$ and $\psi_f(G) = \max \bigcap_{i=1}^{n} \mathrm{Fall}(G_i)$.*

# 6 Restriction of fall $t$-colorings of a graph into proper $t$-colorings of graphs in a specified set

Now we prove that fall $k$-colorings of a graph can be reduced into proper $k$-colorings of graphs in a specified set.

Let $G$ be a graph and $1 \leq t \leq \delta(G) + 1$ be a fixed natural number. For each $v \in V(G)$, choose $t - 1$ arbitrary elements of $N_G(v)$ and join these $t - 1$ vertices to each other and name the new graph $H$. Let $\widehat{G_t}$ be the set of all graphs $H$ constructed like this.

**Theorem 10.** *For each* $1 \le t \le \delta(G)+1$, $t \in \mathrm{Fall}(G)$ *iff* $t \in \{\chi(H) |\ H \in \widehat{G_t}\}$. *Specially,* $\mathrm{Fall}(G) = \bigcup_{i=1}^{\delta(G)+1} (\ \{\chi(H) |\ H \in \widehat{G_i}\} \bigcap \{i\}\ )$.

**Proof.** Let $1 \le t \le \delta(G)+1$. If $t \in \{\chi(H) |\ H \in \widehat{G_t}\}$, then, there exists a graph $H$ in $\widehat{G_t}$ such that $\chi(H) = t$ and there exists a $t$-coloring $f$ of $H$. This coloring $f$ of $H$, is obviously a fall $t$-coloring of $G$ and therefore, $t \in \mathrm{Fall}(G)$. Conversely, if $t \in \mathrm{Fall}(G)$, then, there exists a fall $t$-coloring $g$ of $G$. For each $v \in V(G)$, there exist $t-1$ elements of $N_G(v)$ such that the set of their colors and the color of $v$ is equal to $[t]$, join all of them to each other to construct a new graph $T$ in $\widehat{G_t}$. The fall $t$-coloring $g$ of $G$ is obviously a $t$-coloring of $T$, also $\omega(T) \ge t$, thus, $\chi(T) = t$ and $t \in \{\chi(H) |\ H \in \widehat{G_t}\}$. The second part of the theorem follows immediately.

■

Restricting this theorem into $r$-regular graphs and $t = r+1$, yields a beautiful proposition of [4] but in different terminologies.

**Proposition 3.** *For each $r$-regular graph $G$,* $r+1 \in \mathrm{Fall}(G)$ *iff* $\chi(G^{(2)}) = r+1$, *where* $G^{(2)} = (\ V(G)\ ,\ \{\ \{x,y\}\ |\ x,y \in V(G),\ x \ne y,\ d_G(x,y) \le 2\ \}\ )$.

## 7 Fall Colorings of Mycielskian of graphs

Let $G := (\ \{x_1,\dots,x_n\}\ ,\ E(G)\ )$ be a graph. The Mycielskian of $G$ ( denoted by $M(G)$ ) is a graph with $2n+1$ vertices $x_1,\dots,x_n,y_1,\dots,y_n,z$ with edge set $E(G) \bigcup \{\ \{y_i,x_j\}\ |\ i,j \in [n],\ \{x_i,x_j\} \in E(G)\ \} \bigcup \{\ \{z,y_i\}\ |\ i \in [n]\ \}$.

For example, $M(K_2)$ is $C_5$. We know that $\mathrm{Fall}(M(K_2)) = \mathrm{Fall}(C_5) = \emptyset$. Now we prove that for each graph $G$, $\mathrm{Fall}(M(G)) = \emptyset$.

**Theorem 11.** *For each graph $G$,* $\mathrm{Fall}(M(G)) = \emptyset$.

**Proof.** If $E(G) = \emptyset$, then, $M(G)$ has at least one isolated vertex and also at least one edge. Therefore, $\mathrm{Fall}(M(G)) = \emptyset$. Now we prove the theorem for the case $E(G) \ne \emptyset$. If $E(G) \ne \emptyset$ and $\mathrm{Fall}(M(G)) \ne \emptyset$, then, there exists a fall $k$-coloring $f$ of $M(G)$ for some $k \in \mathbb{N}$. Since $E(G) \ne \emptyset$, there exists an integer $i_0 \in [n]$ such that $f(x_{i_0}) \ne f(z)$ and since for each $j \in [n]$, $f(y_j) \ne f(z)$ and $f$ is a fall $k$-coloring, there exists $i_1 \in [n]$ such that $x_{i_1} \in N_G(x_{i_0})$ and $f(x_{i_1}) = f(z)$. Since for each $i \in [n]$ with $f(x_i) \ne f(z)$, $N(y_i) \setminus \{z\} \subseteq N(x_i)$, so $f(x_i) \in \{f(y_i), f(z)\}$, on the other hand, $f(x_i) \ne f(z)$, hence, $f(x_i) = f(y_i)$. This immediately shows that each color of $[k]$ appears on the neighborhood of $y_{i_1}$, which is a contradiction. Hence, $\mathrm{Fall}(M(G)) = \emptyset$.

■

## 8 Fall colorings of complement of bipartite graphs

Complements of bipartite graphs are very interesting graphs, because in each proper $k$-coloring, the cardinality of each color class is at most 2. The following theorem characterizes all fall colorings of this type of graphs.

**Theorem 12.** *Let $G$ be a bipartite graph. Then,* $\mathrm{Fall}(G^c) \subseteq \{\ \chi(G^c)\ \}$*. Besides, it is polynomial to decide whether or not* $\mathrm{Fall}(G^c) = \{\ \chi(G^c)\ \}$ *.*

**Proof.** If $\mathrm{Fall}(G^c) \neq \emptyset$, then, $\exists k \in \mathrm{Fall}(G^c)$. Suppose that $f$ is a fall $k$-coloring of $G^c$. Obviously, each color class of $f$ is either of the form $\{x\}$ or of the form $\{y, z\}$ such that $y \in A$ and $z \in B$. A color class is of the form $\{x\}$ iff $x$ is an isolated vertex of the graph $G$. Therefore, the set of color classes of $f$ is the union of $\{\ \{x\} \mid x \in V(G),\ deg_G(x) = 0\ \}$ and the set of edges of a perfect matching of the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$, also, $k = |V(G)| - \frac{1}{2}|\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}|$. Therefore, $\mathrm{Fall}(G^c) \subseteq \{\ |V(G)| - \frac{1}{2}|\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}|\ \}$. Besides, if $\mathrm{Fall}(G^c) \neq \emptyset$, then, the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$ has a perfect matching, in this case, obviously, $\chi(G^c) = |V(G)| - \frac{1}{2}|\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}|$, and consequently, $\mathrm{Fall}(G^c) = \{\ \chi(G^c)\ \}$. Therefore, for each bipartite graph $G$, $\mathrm{Fall}(G^c) \subseteq \{\ \chi(G^c)\ \}$. We know that if $\mathrm{Fall}(G^c) \neq \emptyset$, then, the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$ has a perfect matching. Conversely, if the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$ has a perfect matching, then, the union of $\{\ \{x\} \mid x \in V(G),\ deg_G(x) = 0\ \}$ and the edge set of each perfect matching of the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$ is the set of color classes of a fall ( $|V(G)| - \frac{1}{2}|\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}|$ )-coloring of $G^c$ and therefore, $\mathrm{Fall}(G^c) \neq \emptyset$. Accordingly, $\mathrm{Fall}(G^c) = \{\ \chi(G^c)\ \}$ iff $\mathrm{Fall}(G^c) \neq \emptyset$ iff the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$ has a perfect matching. Since the problem of deciding whether or not the induced subgraph of $G$ on $\{\ x \mid x \in V(G),\ deg_G(x) > 0\ \}$ has a perfect matching, is a polynomial time problem, thus, it is polynomial time to decide whether or not $\mathrm{Fall}(G^c) = \{\ \chi(G^c)\ \}$. ■

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